

THE BAUM-CONNES ASSEMBLY MAP AND THE GENERALIZED BASS CONJECTURE

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INTRODUCTION

In the early 1980's, P. Baum and A. Connes defined an assembly map

$$(0.1) \quad \mathcal{A}_*^{G,a} : KK_*^G(C(\underline{EG}), \mathbb{C}) \rightarrow K_*^t(C_r^*(G))$$

where G denotes a locally compact group, \underline{EG} the classifying space for proper G -actions, $C(\underline{EG})$ the G -algebra of complex-valued functions on \underline{EG} vanishing at infinity, and $KK_*^G(C(\underline{EG}), \mathbb{C})$ the G -equivariant KK -groups of (\underline{EG}) with coefficients in \mathbb{C} , while $K_*^t(C_r^*(G))$ represents the topological K -groups of the reduced C^* -algebra of G . The original details of this map appeared (a few years later) in [BC1] and [BC2], with further elaborations in [BCH]. As shown in [BC3], when G is discrete the left-hand side admits a Chern character which may be represented as

$$ch_*^{BC}(G) : KK_*^G(C(\underline{EG}), \mathbb{C}) \rightarrow \bigoplus_{x \in \text{fin}(<G>)} H_*(BG_x; \mathbb{C}) \otimes HPer_*(\mathbb{C})$$

where $\text{fin}(<G>)$ is the set of conjugacy classes of G corresponding to elements of finite order, G_x the centralizer of g in G where $x = <g>$, and $HPer_*(\mathbb{C})$ the periodic cyclic homology of \mathbb{C} . Note that $H_*(BH; \mathbb{C}) \otimes HPer_*(\mathbb{C})$ are simply the 2-periodized complex homology groups of BH , and (via the classical Atiyah-Hirzebruch Chern character) can be alternatively viewed as the complexified K -homology groups of BH . Upon complexification, the map $ch_*^{BC}(G)$ is an isomorphism. The original construction of Baum and Connes $\mathcal{A}_*^{G,a}$ was analytical. Motivated by the need to construct a homotopical analogue to their map, we constructed an assembly map in [O1] which we will denote here as

$$\mathcal{A}_*^{G,h} \otimes \mathbb{C} : H_*\left(\coprod_{x \in \text{fin}(<G>)} BG_x; \mathbf{K}(\mathbb{C})\right) \otimes \mathbb{C} \rightarrow K_*^t(C_r^*(G)) \otimes \mathbb{C}$$

where $\mathbf{K}(\mathbb{C})$ denotes the 2-periodic topological K -theory spectrum of \mathbb{C} . The construction of this map amounted to an extension of the classical assembly map constructed in [L] which was designed to take into account the contribution coming from the conjugacy classes of finite order. The two

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

essential features of $\mathcal{A}_*^{G,h} \otimes \mathbb{C}$, shown in [O1], were (i) it factors through $K_*^t(\mathbb{C}[G]) \otimes \mathbb{C}$ (where $K_*^t(\mathbb{C}[G])$ denotes the Bott-periodized topological K -theory of the complex group algebra, topologized with the fine topology), and (ii) the composition of $\mathcal{A}_*^{G,h} \otimes \mathbb{C}$ with the complexified Chern-Connes-Karoubi-Tillmann character $ch_*^{CK} : K_*(\mathbb{C}[G]) \otimes \mathbb{C} \rightarrow HC_*(\mathbb{C}[G])$ was effectively computable (see below). What we did not do in [O1] was show that $\mathcal{A}_*^{G,a} \otimes \mathbb{C}$ and $\mathcal{A}_*^{G,h} \otimes \mathbb{C}$ agree. Since this initial work, there have been numerous extensions and reformulations of the Baum-Connes assembly map, as well as of the original Baum-Connes conjecture, which states that the map in (0.1) is an isomorphism. These extensions typically are included under the umbrella term “Isomorphism Conjecture”, (formulated for both algebraic and topological K -theory; cf. [DL], [FJ], [LR]). Thanks to [HP], we now know that the different formulations of these assembly maps (e.g., homotopy-theoretic vs. analytical) agree.

Abbreviating $KK_*^G(C(\underline{EG}), \mathbb{C})$ as $K_*^G(\underline{EG})$ (read: the equivariant K -homology of the proper G -space \underline{EG}), our main result is

Theorem 1. *There is a commuting diagram*

$$\begin{array}{ccc} K_*^G(\underline{EG}) & \xrightarrow{\mathcal{A}_*^{G,DL}} & K_*^t(\mathbb{C}[G]) \\ \downarrow ch_*^? & & \downarrow ch_*^{CK} \\ HC_*^{fin}(\mathbb{C}[G]) & \xrightarrow{\quad} & HC_*(\mathbb{C}[G]) \end{array}$$

where $\mathcal{A}_*^{G,DL}$ is the homotopically defined assembly map of [DL], $^{fin}HC_*(\mathbb{C}[G]) := \bigoplus_{x \in \text{fin}(\langle G \rangle)} HC_*(\mathbb{C}[G])_x \cong \bigoplus_{x \in \text{fin}(\langle G \rangle)} H(BG_x; \mathbb{C}) \otimes HC_*(\mathbb{C})$ is the elliptic summand of $HC_*(\mathbb{C}[G])$ [JOR], the lower horizontal map is the obvious inclusion, and the Chern character $ch_*^?$ becomes an isomorphism upon complexification for $* \geq 0$.

Let β denote a bounding class, (G, L) a discrete group equipped with a word-length, and $H_{\beta,L}(G)$ the rapid decay algebra associated with this data [JOR]. We write $K_*^t(H_{\beta,L}(G))$ for the Bott-periodic topological K -theory of the topological algebra $H_{\beta,L}(G)$. The Baum-Connes assembly map for $H_{\beta,L}(G)$ is defined to be the composition

$$(BC) \quad \mathcal{A}_*^{G,\beta} : K_*^G(\underline{EG}) \xrightarrow{\mathcal{A}_*^{G,DL}} K_*^t(\mathbb{C}[G]) \rightarrow K_*^t(H_{\beta,L}(G))$$

where the second map is induced by the natural inclusion $\mathbb{C}[G] \hookrightarrow H_{\beta,L}(G)$. In [JOR], we conjectured that the image of $ch_* : K_*^t(H_{\beta,L}(G)) \rightarrow HC_*^t(H_{\beta,L}(G))$ lies in the elliptic summand $^{fin}HC_*^t(H_{\beta,L}(G))$ (conjecture β -SrBC). As the inclusion $\mathbb{C}[G] \hookrightarrow H_{\beta,L}(G)$ sends $^{fin}HC_*(\mathbb{C}[G])$ to $^{fin}HC_*^t(H_{\beta,L}(G))$, naturality of the Chern character ch_*^{CK} and Theorem 1 implies

Corollary 2. *If $\mathcal{A}_*^{G,\beta}$ is rationally surjective, then β -SrBC is true.*

Since going down and then across is rationally injective, we also have (compare [O1])

Corollary 3. *The assembly map $\mathcal{A}_*^{G,DL} \otimes \mathbb{Q}$ is injective for all discrete groups G .*

We do not claim any great originality in this paper. In fact, Theorem 1, although not officially appearing in print before this time, has been a “folk-theorem” known to experts for many years. The connection between the Baum-Connes Conjecture (more precisely a then-hypothetical Baum-Connes-type Conjecture for $\mathbb{C}[G]$) and the stronger Bass Conjecture for $\mathbb{C}[G]$ discussed in [JOR] was noted by the author in [O2].

There is some overlap of this paper with the results presented in [Ji]. A special case of Theorem 1 (for $*$ = 0 and $\mathbb{C}[G]$ replaced by the ℓ^1 -algebra $\ell^1(G)$) appeared as the main result of [BCM].

PROOF OF THEOREM 1

We use the notation $F_*^{fin}(\mathbb{C}[G])$ to denote the elliptic summand $\bigoplus_{x \in fin(<G>)} F_*(\mathbb{C}[G])_x$ of $F_*(\mathbb{C}[G])$ where $F_*(-) = HH_*(-), HN_*(-), HC_*(-)$ or $HPer_*(-)$. To maximize consistency with [LR], we write \mathbf{S} for the (unreduced) suspension spectrum of the zero-sphere S^0 , $\mathbf{HN}(R)$ resp. $\mathbf{HH}(R)$ the Eilenberg-MacLane spectrum whose homotopy groups are the negative cyclic resp. Hochschild homology groups of the discrete ring R , and $\mathbf{K}^a(R)$ the non-connective algebraic K -theory spectrum of R , with $K_*^a(R)$ representing its homotopy groups. By [LR, diag. 1.6] there is a commuting diagram

$$(1.1) \quad \begin{array}{ccccc} H_*^G(\underline{EG}; \mathbf{S}) & \xrightarrow{\quad} & K_*^a(\mathbb{Z}[G]) & & \\ \downarrow & & \downarrow NTr_* & & \\ H_*^G(\underline{EG}; \mathbf{HN}(\mathbb{Z})) & \xrightarrow{\cong} & HN_*^{fin}(\mathbb{Z}[G]) & \xrightarrow{\quad} & HN_*(\mathbb{Z}[G]) \\ \downarrow & & \downarrow & & \downarrow h_* \\ H_*^G(\underline{EG}; \mathbf{HH}(\mathbb{Z})) & \xrightarrow{\cong} & HH_*^{fin}(\mathbb{Z}[G]) & \xrightarrow{\quad} & HH_*(\mathbb{Z}[G]) \end{array}$$

where the top horizontal map is the composition

$$H_*^G(\underline{EG}; \mathbf{S}) \rightarrow H_*^G(\underline{EG}; \mathbf{K}^a(\mathbb{Z})) \xrightarrow{\mathcal{A}^{G, DL}} K_*(\mathbb{Z}[G])$$

referred to as the restricted assembly map for the algebraic K -groups of $\mathbb{Z}[G]$. The other two horizontal maps are the assembly maps for negative cyclic and Hochschild homology respectively. The upper left-hand map is induced by the map from the sphere spectrum to the Eilenberg-MacLane spectrum \mathbf{HN} , which may be expressed as the composition of spectra $\mathbf{S} \rightarrow \mathbf{K}^a(\mathbb{Z}) \rightarrow \mathbf{HN}$. By [LR], the composition on the left is a rational equivalence.

Let \mathbb{C}^δ denote the complex numbers \mathbb{C} equipped with the discrete topology. Tensoring with \mathbb{C} and combined with the inclusion of group algebras $\mathbb{Z}[G] \hookrightarrow \mathbb{C}^\delta[G]$, (1.1) yields the commuting diagram

$$(1.2) \quad \begin{array}{ccc} H_*^G(\underline{EG}; \mathbb{Q}) \otimes \mathbb{C} & \longrightarrow & K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} \\ \downarrow \cong & & \downarrow NTr_* \\ HN_*^{fin}(\mathbb{C}[G]) & \xrightarrow{\quad} & HN_*(\mathbb{C}[G]) \end{array}$$

Next, we consider the transformation from algebraic to topological K -theory, induced by the map of group algebras $\mathbb{C}^\delta[G] \rightarrow \mathbb{C}[G]$ which is the identity on elements. By the results of [CK], [W] and [T], there is a commuting diagram

$$(1.3) \quad \begin{array}{ccc} K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\ \downarrow NTr_* & & \downarrow ch_*(\mathbb{C}[G]) \\ HN_*(\mathbb{C}[G]) & \longrightarrow & HPer_*(\mathbb{C}[G]) \end{array}$$

where $ch_*(\mathbb{C}[G])$ is the Connes-Karoubi Chern character for the fine topological algebra $\mathbb{C}[G]$, and the bottom map is the transformation from negative cyclic to periodic cyclic homology.

We can now consider our main diagram

$$\begin{array}{ccccccc}
 H_*^G(\underline{EG}; \mathbb{C}) \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^a(\mathbb{C}^\delta[G]) \otimes \mathbb{C} \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\
 \downarrow & & \downarrow & & \downarrow \scriptstyle{ch_*(\mathbb{C}[G]) \otimes ch_*(\mathbb{C}\{\{id\}\})} & & \downarrow \scriptstyle{ch_*(\mathbb{C}[G])} \\
 (1.4) \quad HN_*^{fin}(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) & \longrightarrow & HN_*(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) & \longrightarrow & HPer_*(\mathbb{C}[G]) \otimes HPer_*(\mathbb{C}) & \xrightarrow{\cong} & HPer_*(\mathbb{C}[G]) \\
 \downarrow & & & & \uparrow & & \uparrow \\
 HPer_*^{fin}(\mathbb{C}[G]) \otimes K_*(\mathbb{C}) & \xrightarrow{\cong} & HPer_*^{fin}(\mathbb{C}[G]) \otimes HPer_*(\mathbb{C}) & \xrightarrow{\cong} & HPer_*^{fin}(\mathbb{C}[G]) & &
 \end{array}$$

The top left square commutes by (1.2), and the middle top square commutes by (1.3). The upper right square commutes by virtue of the fact that the Connes-Karoubi-Chern character is a homomorphism of graded modules, which maps the $K_*^t(\mathbb{C})$ -module $K_*^t(\mathbb{C}[G])$ to the $HPer_*(\mathbb{C})$ -module $HPer_*(\mathbb{C}[G])$, with the map of base rings induced by isomorphism $ch_*(\mathbb{C}\{\{id\}\}) : K_*^t(\mathbb{C}) \otimes \mathbb{C} \xrightarrow{\cong} HPer_*(\mathbb{C})$. The lower left square commutes trivially, while the lower right commutes by the naturality of the inclusion $HPer_*^{fin}(\mathbb{C}[G]) \hookrightarrow HPer_*(\mathbb{C}[G])$ with respect to the module structure over $HPer_*(\mathbb{C})$. Summarizing, we get a commuting diagram

$$\begin{array}{ccc}
 H_*^G(\underline{EG}; \mathbb{C}) \otimes K_*(\mathbb{C}) & \longrightarrow & K_*^t(\mathbb{C}[G]) \otimes \mathbb{C} \\
 \downarrow \scriptstyle{\cong} & & \downarrow \scriptstyle{ch_*(\mathbb{C}[G])} \\
 (1.5) \quad HPer_*^{fin}(\mathbb{C}[G]) & \longrightarrow & HPer_*(\mathbb{C}[G]) \\
 \downarrow & & \downarrow \\
 HC_*^{fin}(\mathbb{C}[G]) & \longrightarrow & HC_*(\mathbb{C}[G])
 \end{array}$$

where the bottom square is induced by the transformation $HPer_*(-) \rightarrow HC_*(-)$, which respects the summand decomposition indexed on conjugacy classes. Restricted the elliptic summand yields the map $HPer_*^{fin}(\mathbb{C}[G]) \rightarrow HC_*^{fin}(\mathbb{C}[G])$ which is an isomorphism for $* \geq 0$, implying the result stated in Theorem 1.

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